Physics 214 UCSD

Lecture 6

- Outlook for remainder of quarter
- Halzen & Martin Chapter 3
- Start of Halzen & Martin Chapter 4

Outlook for remaining Quarter

- From now on I will follow H&M more closely.
- We’ll basically cover chapters 3,4,5,6
  - A lot of this should be a review of things you have seen already either in advanced QM or intro QFT.
  - Accordingly, I’ll be brief at times, and expect you to read up on it as needed !!!
- Then skip chapter 7.
- Then parts of 8,9,10, and 11, where I am not yet sure as to the order I’ll do them in.
- Some of this we won’t get to until next quarter.

(non-)relativistic Schroedinger Eq.

- Nonrelativistic

  $E = p^2/(2m)$

- Relativistic

  $E^2 = p^2+m^2$

In both cases we replace:

$$E \rightarrow i\frac{\partial}{\partial t}$$

$$p \rightarrow -i\nabla$$

$$\left(i\frac{\partial}{\partial t} + \frac{\nabla^2}{2m}\right)\psi = 0$$

$$-\frac{\partial^2}{\partial t^2}\phi = \left(-\nabla^2 + m^2\right)\phi$$

$$\left(i\frac{\partial}{\partial t}+\frac{\nabla^2}{2m}\right)\psi=0$$

$$-\frac{\partial^2}{\partial t^2}\phi=\left(-\nabla^2+m^2\right)\phi$$

In both cases we have plane wave solutions as:

$$\phi\left(t,\vec{x}\right)=Ne^{-ip_\mu x^\mu}$$

Covariant Notation

$A^\mu = (A^0, \mathbf{A})$; $A_\mu = (A^0, -\mathbf{A})$

$A^\mu B_\mu = A^0 B^0 - \mathbf{AB}$

The derivative 4-vector is given by:

$$\partial^\mu = \left(\frac{\partial}{\partial t}, -\nabla\right)$$

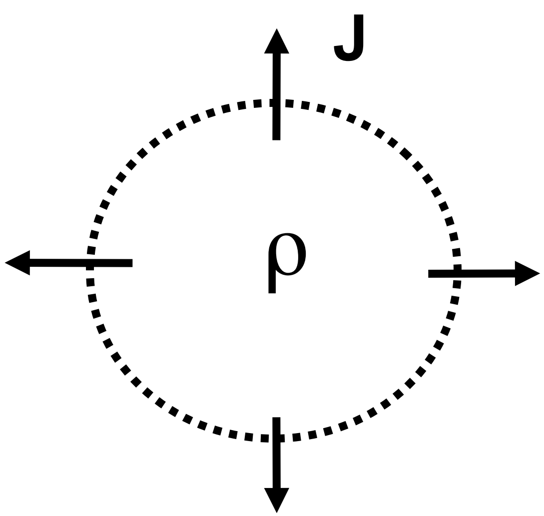
$$\partial_\mu = \left(\frac{\partial}{\partial t}, \nabla\right)$$

With: $\square^2 = \partial_\mu \partial^\mu$

We then get the Klein-Gordon Equation as: $(\square^2 + m^2)\,\phi = 0$

Continuity Equation

- For scattering, we need to understand the probability density flux **J**, as well as the probability density ρ.
- Conservation of probability leads to:



$$-\frac{\partial}{\partial t}\int_V \rho dV = \int_S \vec{j}\cdot\hat{n}ds = \int_V \nabla J dV$$

$$\Rightarrow \boxed{\nabla J + \frac{\partial \rho}{\partial t} = 0}$$

(non-)relativistic Continuity Eq.

- Nonrelativistic

$$\rho = |\psi|^2$$

$$J = \frac{-i}{2m}\left(\psi^* \nabla \psi - \psi \nabla \psi^*\right)$$

(units: $\hbar$ = 1)

- Relativistic

$$\rho = i\left(\phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t}\right)$$

$$J = -i\left(\phi^* \nabla \phi - \phi \nabla \phi^*\right)$$

(units: $\hbar$ = c = 1)

Covariant Notation

$$j^{\mu} = \left(\rho, \vec{j}\right)$$ Transforms like a 4-vector

$$\partial^{\mu} j_{\mu} = 0$$ Covariant continuity equation

For the plane wave solutions we find:

$$\left.\begin{array}{l} \rho = 2E\,|N|^2 \\ \vec{j} = 2\vec{p}\,|N|^2 \end{array}\right\} J^{\mu} = 2p^{\mu}\,|N|^2$$

Why $\rho \propto E$?

ρd^3x = constant under lorentz transformations

However, d^3x gets lorentz contracted.

Therefore, ρ must transform time-like, i.e. dilate.

$$d^3x \rightarrow d^3x \cdot \sqrt{1-v^2}$$

$$\rho \rightarrow \rho \big/ \sqrt{1-v^2}$$

Energy Eigenvalues of K.G. Eq.

$$(\Box^2 + m^2) \phi = 0$$

Or

$$E^2 = p^2 + m^2$$

$$\Downarrow$$

$$E = \pm\sqrt{p^2 + m^2}$$

Positive and negative energy solutions !

Feynman-Stueckelberg Interpretation

- Positive energy particle moving forwards in time.
- Negative energy antiparticles moving backwards in time.

$\Rightarrow$ Absorption of positron with -E is the same as emission of electron with +E.

$\Rightarrow$ In both cases charge of system increases while energy decreases.

Encourage you to read up on this in chapters 3.4 & 3.5 of H&M.

- Will get back to discussing negative energy solutions after we understand scattering in a potential.
- Will use scattering in a potential to discuss perturbation theory.
  - Assume potential is finite in space.
  - Incoming and outgoing states are free-particle solutions “far enough” away from potential.
  - Assume V is a small perturbation throughout such that free particle, i.e, initial state plane wave, is a meaningful approximation.

Nonrelativistic Perturbation Theory

- Assume we know the complete set of eigenstates of the free-particle Schroedinger Equation:

$$H\phi_n = E_n\phi$$

$$\int_{Vol} \phi_n^* \phi_m d^3x = \delta_{nm}$$

- Now solve Schroedinger Eq. in the presence of a small time-dependent perturbation $V(\boldsymbol{x},t)$:

$$(H + V)\psi = i\frac{\partial\psi}{\partial t}$$

Any solution can be expressed as:

$$\psi = \sum_n a_n(t)\phi_n(x)e^{-iE_nt}$$

Plug this into Sch.Eq. and you get:

$$i\sum_n \frac{da_n(t)}{dt}\phi_n(x)e^{-iE_nt} = V(x,t)\sum_n a_n(t)\phi_n(x)e^{-iE_nt}$$

Multiply both sides by: $-ie^{iE_ft}\phi_f^*$

$$\frac{da_f(t)}{dt} = -i\sum_n a_n(t)e^{i(E_f-E_n)t}\int \phi_f^* V\phi_n d^3x$$

What does this mean?

$$\frac{da_f(t)}{dt} = -i\sum_n a_n(t)e^{i(E_f - E_n)t}\int \phi_f^* V \phi_n d^3x$$

Let's think of it as scattering of an incoming eigenstate *i* into an outgoing eigenstate *f*.

Initial and final states are eigenstates because
V is small, limited in space,
and experienced only for finite amount of time..

Assume V is small and "seen" for only a finite amount of time.

- At times long ago, the system is in eigenstate i of the free hamiltonian because it's far away from V.
- At times far in the future, the system is in eigenstate f of the free hamiltonian because it's far away from V.
- After integration over time, we thus get:

$$\frac{da_f}{dt} = -ie^{-i(E_f - E_i)t} \int d^3x \phi_f^* V \phi_i$$ ☜ starting point: $i \rightarrow f$

$$V_{fi} \equiv \int d^3x \phi_f^* V(x) \phi_i$$ ☜ Assume (for now) V is time independent

$$T_{fi} \equiv a_f = -iV_{fi} \int dt e^{-i(E_f - E_i)t} = \boxed{-2\pi i V_{fi} \delta(E_f - E_i)}$$

Result of time integration.

Meaning of $T_{fi} = -2\pi i V_{fi} \delta(E_f - E_i)$

- δ-function guarantees energy conservation.

$\Rightarrow$ Uncertainty principle guarantees that T_{fi} is meaningful only as $t \to \infty$.

- We thus define a more meaningful quantity W, the ***"transition probability per unit time"*** by dividing with t, and then letting $t \to \infty$.

$$W = \lim_{t \to \infty} \frac{\left|T_{fi}\right|^2}{t} = 2\pi \left|V_{fi}\right|^2 \delta(E_f - E_i)$$

Aside

$$W = \lim_{t->\infty} \frac{\left|T_{fi}\right|^2}{t} = \lim_{t->\infty} 2\pi \frac{\left|V_{fi}\right|^2}{t} \delta(E_f - E_i) \int_{-t/2}^{t/2} dt e^{-i(E_f - E_i)t}$$

The δ-function from the first integral guarantees that the second integral is *t*, and thus cancels with the 1/*t*, making the limit calculation trivial.

$$T_{fi} T_{fi}^* = (-i V_{fi} \int dt e^{-i(E_f - E_i)t})(-i V_{fi} \int dt e^{-i(E_f - E_i)t})^* =$$

$$\left|T_{fi}\right|^2 = 2\pi \left|V_{fi}\right|^2 \delta(E_f - E_i) \int dt e^{-i(E_f - E_i)t}$$

Physically meaningful quantities

- The transition probability per unit time, *W*, becomes physically meaningful once you integrate over a set of initial and final states.
- Though typically, we start with a specific initial and a set of final states:

$$W_{fi} = 2\pi \int dE_f \rho(E_f) \left|V_{fi}\right|^2 \delta(E_f - E_i)$$

$$W_{fi} = 2\pi \left|V_{fi}\right|^2 \rho(E)$$

☞ ***Fermi's Golden Rule***

Fermi’s Golden Rule

- We have found Fermi’s Golden Rule as the leading order in perturbation theory.
- This begs the question, what’s the next order, and how do we get it?
- In our lowest order approximation, we scattered from an initial state *i* to a final state *f*.
- The obvious improvement is to allow for double scattering from *i* to any *n* to *f*, and sum over all *n*.

Second Order

$$a_n = -i\int dt e^{i(E_n - E_i)} V_{ni}$$

$$\frac{da_f(t)}{dt} = LO - i\sum_{n\neq i} a_n(t) e^{i(E_f - E_n)t} V_{fn}$$

$$W_{fi} = 2\pi \left| V_{fi} + \sum_{n\neq i} V_{fn} \frac{1}{E_i - E_n + i\varepsilon} V_{ni} \right|^2 \rho(E_i)$$

What have we learned?

- For each interaction vertex we get a vertex factor V_{fi} .
- For the propagation via an intermediate state we gain a “propagator” factor $1/(E_i-E_n)$.
- The intermediate state is virtual, and thus does not require energy conservation.
- However, energy is conserved between initial and final state.

$$W_{fi} = 2\pi \left| V_{fi} + \sum_{n \neq i} V_{fn} \frac{1}{E_i - E_n + i\varepsilon} V_{ni} \right|^2 \rho(E_i)$$

Time dependent perturbation

- Now restore time dependence, $V = V(t)$
- Assume harmonic, $V_{fi}(t) = V_{fi}\, e^{-i\omega t}$
- Now

$$\frac{da_f}{dt} = -ie^{i(E_f - E_i - \omega)t} V_{fi}$$

- Likewise for other intermediate states
- Energy conservation δ-function is now

$$\delta(E_f - E_i - \omega)$$

Let’s play some games with this formalism

Photon absorption by Particle vs Antiparticle

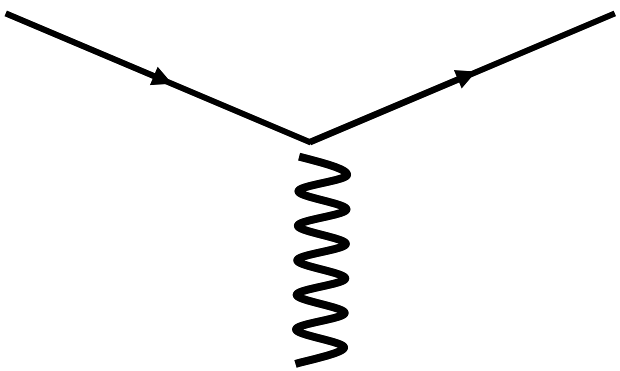
- Particle scatter in field

$$p_i = (E_1, \overrightarrow{p_1})$$

$$p_f = (E_2, \overrightarrow{p_2})$$

$$T_{fi} \approx \int dt \phi_f^* V \phi_i \approx \int dt e^{iE_2 t} e^{-i\omega t} e^{-iE_1 t}$$

$$\approx \delta(E_2 - (\omega + E_1))$$



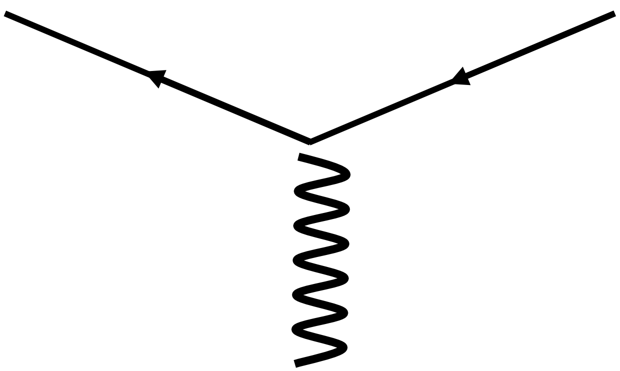
- Antipart. scatter in field

$$p_i = (-E_2, -\overrightarrow{p_2})$$

$$p_f = (-E_1, -\overrightarrow{p_1})$$

$$T_{fi} \approx \int dt \phi_f^* V \phi_i \approx \int dt e^{i(-E_1)t} e^{-i\omega t} e^{-i(-E_2)}$$

$$\approx \delta(E_2 - (\omega + E_1))$$



Particle and antiparticle have the same interaction with EM field.

Pair Creation from this potential

$$p_i = (-E_1, -\overrightarrow{p_1})$$

$$p_f = (E_2, \overrightarrow{p_2})$$

$$T_{fi} \approx \int dt \phi_f^* V \phi_i \approx \int dt e^{iE_2 t} e^{-i\omega t} e^{-i(-E_1)t}$$

$$\approx \delta(E_2 + E_1 - \omega)$$

Energy is conserved as it should be.

This wave function formalism is thus capable of describing particles, antiparticles, and pair production.

“Rules”

- Time goes from left to right
- Antiparticles get arrow that is backwards in time.
- “Incoming” and “outgoing” is defined by how the arrows point to the vertex.
- Antiparticles get negative energy assigned.