

# Spring Quarter 2015

## UCSD Physics 214

### Homework 3

This Homework assignment covers a mix of problems from symmetries and neutrino oscillations. For the latter, you might find the following general hints useful:

#### General hints

A useful trick to simplify neutrino oscillation problems is to make use of unitarity, for example:

$$U_{ei}U_{ej}^* + U_{\mu i}U_{\mu j}^* + U_{\tau i}U_{\tau j}^* = \delta_{ij}$$

Then for  $i \neq j$  multiply by  $U_{ei}^*U_{ej}$  to get

$$|U_{ei}U_{ej}^*|^2 + U_{ei}^*U_{ej}(U_{\mu i}U_{\mu j}^* + U_{\tau i}U_{\tau j}^*) = 0$$

$$U_{ei}^*U_{ej}(U_{\mu i}U_{\mu j}^* + U_{\tau i}U_{\tau j}^*) = -|U_{ei}U_{ej}^*|^2$$

Another useful trick is

$$\sum_i U_{\alpha i}^*U_{\beta i} = \delta_{\alpha\beta}$$

Take the square of this:

$$\sum_{ij} U_{\alpha i}^*U_{\beta i}U_{\alpha j}U_{\beta j}^* = \delta_{\alpha\beta}$$

$$\sum_i U_{\alpha i}^*U_{\beta i}U_{\alpha i}U_{\beta i}^* + 2\sum_{i<j} U_{\alpha i}^*U_{\beta i}U_{\alpha j}U_{\beta j}^* = \delta_{\alpha\beta}$$

Rearranging:

$$\sum_i U_{\alpha i}^*U_{\beta i}U_{\alpha i}U_{\beta i}^* = \delta_{\alpha\beta} - 2\sum_{i<j} U_{\alpha i}^*U_{\beta i}U_{\alpha j}U_{\beta j}^*$$

You can do similar things interchanging the roles of rows and columns.

1. The amplitude for  $\nu_\alpha \rightarrow \nu_\beta$  in the general case of N neutrino generations is given by equation (10) in hep-ph/0506165.
  - a. Show that the probability for  $\nu_\alpha \rightarrow \nu_\beta$  oscillations is given by equation (11) in the same paper. If you can not obtain this result, just take it as a given and proceed.
  - b. If the neutrinos are Majorana particles, the neutrino mixing matrix includes two extra phases, as shown in equation (40) of the same paper. Verify that these Majorana phases have no effect on the oscillation probability of equation (11).
  - c. As discussed on page 4 of this paper, there is CP-violation in neutrino oscillations if the imaginary part  $\Im(U_{\alpha i}^*U_{\beta i}U_{\alpha j}U_{\beta j}^*) \neq 0$  for  $\alpha$  not equal  $\beta$ . Use the unitarity of the neutrino mixing matrix to show that for a given pair of flavors,  $\alpha$  not equal  $\beta$ , the quantity  $\Im(U_{\alpha i}^*U_{\beta i}U_{\alpha j}U_{\beta j}^*)$  is the same (up to a sign) for any  $i, j$ . This argument, which is based on unitarity, can be expanded by interchanging the roles of rows and

columns of  $U$  to conclude that for a given pair of  $i, j$  this quantity is independent of  $\alpha$  and  $\beta$  (up to a sign). Thus CP violation is universal in the sense that it is independent of the choice of  $i, j$ ,  $\alpha$  and  $\beta$ . To estimate whether the amount of CP-violation allowed in the Standard Model, after including quark and lepton sector, is sufficient to explain the lack of anti-matter in the universe, we thus need to measure this invariant in both the quark and lepton sector. Use the expression for  $U$  given in equation (40) of this paper (without the Majorana phases) to work out the value of  $\Im(U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^*)$ . Verify that this quantity is proportional to  $s_{12} s_{23} s_{13} \sin\delta$ . This shows that CP-violation needs not only a complex phase  $\delta$  but also non-trivial mixing between all three neutrino states. Mixing provides the CP-invariant phase we talked about in class for the simpler example of CP violation in decay of particles. It explains why measuring the smallest of the three angles determines how difficult the measurement of CP-violation in the neutrino sector will be because it sets the scale. It thus explains the importance of various attempts at measuring  $s_{13}$  worldwide. Note: To save a couple lines of trivial algebra, you might want to take the mixing matrix from 2006 PDG, page 138, rather than this paper. The PDG matrix is for quarks rather than neutrinos, but the functional form is identical.

2. This problem is for you to explore the argument behind the “off-axis” neutrino oscillation experiments like Nova.
  - a. Consider a pion of energy  $E$  in the lab frame. Find the lab energy of the neutrino in the decay:  $\pi^+ \rightarrow \mu^+ \nu_\mu$  as a function of the lab angle  $\theta$  that the emitted neutrino makes with the original flight direction of the charged pion.
  - b. Plot the energy of the neutrino for pion energies between 2GeV and 20GeV in the case  $\theta = 0$  and  $\theta = 15\text{mrad}$ . You should find that the spread in energy for the “off-axis” case is much reduced. This is used by some of the next generation of neutrino oscillation experiments to tune energy and baseline to sit right at the maximum of the oscillations.
3. Reactors are a source of electron anti-neutrinos. The Daya Bay neutrino reactor experiment measured  $\sin^2 2\theta_{13}$  of the neutrino mixing matrix. It measured the probability for electron anti-neutrino disappearance a distance  $L$  from the reactor. Show that this probability is given by:  $P = \sin^2 2\theta_{13} \sin^2 \Delta_{13} + \cos^4 \theta_{13} \sin^2 2\theta_{12} \sin^2 \Delta_{21}$  where  $\theta_{ij}$  are the angles of the neutrino mixing matrix and  $\Delta_{ij} = \frac{1}{4}(m_i^2 - m_j^2)L/E$ . You can make the approximation that  $\Delta_{31} = \Delta_{32}$ . Next, verify that for the typical  $E = 3 \text{ MeV}$  and  $L = 1.5 \text{ Km}$ ,  $\Delta_{31}$  is close to maximal ( $90^\circ$ ) and  $\Delta_{21}$  is quite small. Take  $|\Delta m_{21}^2| = |m_2^2 - m_1^2| = 8.0 \cdot 10^{-5} \text{ eV}^2$  and  $|\Delta m_{31}^2| = |m_3^2 - m_1^2| = 2.4 \cdot 10^{-3} \text{ eV}^2$ .