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Breit Wigners and Form Factors

Abstract

This note summarizes some interesting facts about Breit Wigners and form factors. It was prepared as background for forthcoming notes by CLEOns [1, 2] pursuing analyses that require carefully formulated Breit Wigner amplitudes.

1 Breit Wigners

A Breit Wigner (BW) lineshape is an approximate model for a resonance (unstable particle) propagator in quantum field theory. It is only "exact", and well-defined, for fundamental particles such as the photon, Z^0, and $W^{\pm}$; and even then, only in the limit that the total width is roughly constant over the width of the resonance.

Hadronic resonances, which interact strongly with all other hadrons, are very complex, and the BW form for the propagator is only a (sometimes poor) approximation. Nevertheless, it crops up in a variety of contexts.

1.1 Non-relativistic Breit Wigners — time dependence

Breit Wigners (*cf.*, PDG98 Eqn 35.50) arise in non-relativistic quantum mechanics first via the hand-waving argument that the exponential decay law (a consequence of Fermi's Golden Rule) is encoded in the wave function, and in its Fourier transform:

$$|\psi(t)|^2 = |\psi(0)|^2 e^{-\Gamma t} \quad \Longrightarrow \quad \psi(t) = \psi(0) e^{-\Gamma t/2}, \tag{1}$$

with $\Gamma = 1/\tau$ where τ is the particle lifetime. The energy dependence of the state is thus given by its Fourier transform:

$$\tilde{\psi}(E) = \int \psi(t) e^{iEt} dt \propto \frac{1}{E - M + (i\Gamma/2)}. \tag{2}$$

Reaction rates thus have energy dependence proportional to

$$|\tilde{\psi}(E)|^2 \propto \frac{1}{(E-M)^2 + (\Gamma/2)^2}. \tag{3}$$

1.2 Non-relativistic BW — partial wave phase shifts

More formally, non-relativistic Breit Wigners arise in the context of the partial wave expansion in 3D scattering theory [3]. When an l'th partial wave

$$a_l = e^{i\delta_l} \sin\delta_l = \frac{1}{(\cot\delta_l - i)} \tag{4}$$

experiences a phase shift δ_l near $\pi/2$ at energy E near M, we have resonant scattering. Expanding the partial wave amplitude about the resonance energy yields the Breit Wigner formula:

$$\cot\delta_l(E) \approx -(E-M)\frac{2}{\Gamma}, \quad \Longrightarrow \quad a_l = \frac{\Gamma/2}{M-E-(i\Gamma/2)}. \tag{5}$$

In this context, the BW is clearly an approximation valid only near its resonant peak.

1.3 Relativistic BW — propagators

In relativistic quantum field theory [4] (*i.e.*, in the presence of particle-antiparticle creation), the relativistic BW form (*cf.*, PDG98 Eqn 35.53) is the "two-point function", propagator, or self-energy, for an unstable particle.

The Feynman rules specify the propagator for, *e.g.*, an intermediate massive, unstable, vector or axial-vector resonance:

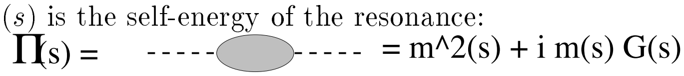
$$BW_{prop} = \frac{i(-g^{\mu\nu} + p^\mu p^\nu/m^2)}{s - m^2(s) - i\sqrt{s}\Gamma(s)}. \tag{6}$$

This form arises when calculating amplitudes associated with Feynman diagrams in terms of weak and strong couplings to resonances. The numerator comes from summing over helicities of the intermediate vector/axial-vector boson resonance:

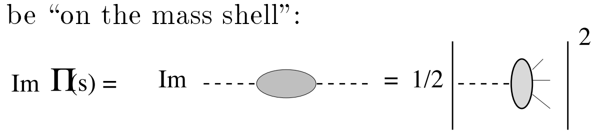
$$\sum_\lambda \varepsilon_\mu^{(\lambda)*}\varepsilon_\nu^{(\lambda)} = -g_{\mu\nu} + p_\mu p_\nu/p^2. \tag{7}$$

The second term in Eqn. 7 ensures that the polarization vectors ε_μ are transverse: $p^\mu\varepsilon_\mu = 0$; but see a further discussion of this point in a later section.

The denominator is the energy dependence of the propagator. More precisely, it should read: $i/(s - \Pi(s))$, where $\Pi(s)$ is the self-energy of the resonance:



The dispersive part is $Re\Pi = m^2(s)$, corresponding to the self-energy Feynman diagrams with intermediate states that are virtual (so that the resonance propagates without decaying), and the absorbtive part, $Im\Pi = \sqrt{s}\Gamma(s)$ corresponding to the self-energy Feynman diagrams that are/can be "on the mass shell":



In general, the function $\Pi(s)$ is very complicated (for hadronic resonances, it is incalculable in any fundamental way). For narrow resonances, it can be approximated by constants: $\Pi(s) = m_R^2 + im_R\Gamma_R$, and thus the propagator $i/\Pi(s)$ is a simple pole in the complex s plane. The Argand diagram (PDG98 section 35.5.3) for such a simple resonance is a circle. Argand diagrams for real hadronic resonances are not even close!

People routinely use $Im\Pi = \sqrt{s}\Gamma(s)$ but approximate $Re\Pi = m_R^2$ to be constant; this turns out to be a pretty good approximation, in particular, for the Z^0.

In this context, if the resonance is narrow, we obtain the standard BW form (*cf.*, PDG98 Eqn 35.52). If the resonance is broad, it will deviate from the standard BW shape, becoming narrower on the leading edge and broader on the trailing edge. A rigorous form can only be obtained if the total width is completely understood; no hadronic resonances have this property!

1.4 The running mass

The dispersive part is $Re\Pi = m^2(s)$, is, in general, not constant. One can define the *pole mass* $m_R^2 \equiv m^2(m_R^2)$ (in terms of itself), and separate ot the variable part:

$$BW = \frac{\Pi(s)}{s - \Pi(s)} = \frac{m_R^2 + \delta m^2(s) + i m_R \Gamma_R(s)}{s - m_R^2 - \delta m^2(s) - i m_R \Gamma_R(s)}, \tag{8}$$

where $\delta m^2(s)$ is the *running* (change in the) pole mass as a function of s. The running mass can be related to the decay width $\Gamma(s)$ via the Kramers-Kronig dispersion relation (*aka* sum rule):

$$m^2(s) = m_R^2 + \frac{1}{\pi}\int_{s_{th}}^{\infty} \frac{m_R \Gamma_{tot}(s\prime)}{(s - s\prime)} ds\prime. \tag{9}$$

In practice, the decay width as $s \to \infty$ is rarely understood well enough to calculate meaningful values for the running mass, and the mass is usually taken to be constant.

For some specific cases, effort has gone into understanding $\Gamma(s)$ in some detail. For example, for $\rho \to \pi\pi$, $\Pi(s)$ is derived [5] from an assumed effective range formula for the P-wave π-π scattering phase shift, assuming $\rho(770)$ meson dominance. This yields

$$BW_{\pi\pi}(s) = \frac{M_\rho^2 + d\, M_\rho \Gamma_\rho}{(M_\rho^2 - s) + f(s) - i\sqrt{s}\Gamma_\rho(s)}, \tag{10}$$

where Γ_ρ denotes $\Gamma_\rho(s = M_\rho^2)$, and

$$f(s) = p_\pi^2(s)\left[h(s) - h(M_\rho^2)\right] - p_0^2\,(s - M_\rho^2)\left.\frac{dh}{ds}\right|_{s=M_\rho^2} \tag{11}$$

$$h(s) = \frac{\Gamma_\rho M_\rho^2}{p_0^3}\,\frac{2p_\pi(s)}{\pi\sqrt{s}}\,\ln\frac{\sqrt{s} + 2p_\pi(s)}{2\,M_\pi}, \tag{12}$$

and d is chosen so as to satisfy the $F_\pi(0) = 1$ condition,

$$d = \frac{3\,M_\pi^2}{\pi\,p_0^2}\ln\frac{M_\rho + 2p_0}{2M_\pi} + \frac{M_\rho}{2\pi\,p_0} - \frac{M_\pi^2 M_\rho}{\pi\,p_0^3}. \tag{13}$$

From CLEO data, we find [6] that this form (extended to include the ρ' state) does indeed fit the $\tau \to \pi\pi\nu$ data well.

1.5 More complicated forms

In general, production or decay into a few mesons cannot be described to high precision using one or even a few Breit Wigners. There will, in general, be multiple intermediate resonant channels, each with different couplings to the initial state and to the final state, and, more importantly, each resonant channel can rescatter into any/all of the others. This rescattering is via the strong interaction, so it can't be neglected relative to a single resonant channel.

Two important examples are: S-wave $\pi - \pi$ scattering near threshold, and $e^+e^- \to \Upsilon(nS) \to B^{(*)}\bar{B}^{(*)}$. In the former case, the scattering is quite strong, so the (many, it is suspected) resonant channels are all broad and all scatter from one to the other until the pair of pions is spit out again. The literature on this subject is vast. I like the work of Törnqvist, and his Unitarized Quark Model [7], which makes use of very general and reasonable assumptions, and a minimum of parameters. The main point here is that the multiple resonances do indeed scatter into each other, and this can be treated by writing the amplitude for, *e.g.*, S-wave $\pi - \pi$ scattering as a matrix of coupled resonant propagators (and including their couplings to the initial and final state, phase space and threshold effects, *etc.*). The matrix must be summed over, and individual propagators (Breit-Wigner like terms) can only be written down after diagonalizing the matrix. The resulting forms are indeed quite complicated, but in the end, it can all be "crudely" mocked up as a single, broad resonance (the σ, or $f_0(400-1200)$).

Another important example (for CLEO) is the coupled channel model, especially the models for $e^+e^- \to \psi(nS) \to D^{(*)}\bar{D}^{(*)}$ and $e^+e^- \to \Upsilon(nS) \to B^{(*)}\bar{B}^{(*)}$. The different quarkonia states mix (via virtual and real intermediate $D\bar{D}$ or $B\bar{B}$ states), and only after diagonalization of the mixing matrix can one determine the physical quarkonium resonances and their branching fractions to $D^{(*)}\bar{D}^{(*)}$ or $B^{(*)}\bar{B}^{(*)}$. The classic work on this subject, required reading for anyone working on quarkonium resonances, is the work by Eichten *et al.* [8]. Here, as well, one takes into account all the different couplings of the resonances to their initial (e^+e^-) and final ($D^{(*)}\bar{D}^{(*)}$ or $B^{(*)}\bar{B}^{(*)}$) final states, and their phase space and threshold properties. Amongst its many successes is its ability to explain the otherwise bizarre pattern of branching fractions of the ψ resonances to $D^{(*)}\bar{D}^{(*)}$ final states. This is the work that guided our search for the peak in the BB^* cross section when CLEO II turned on.

2 Breit Wigner normalizations

There are several approaches to the normalization of the Breit Wigner lineshape, useful for different purposes. Here are some different ways to do it:

- BW as a propagator. In this case, one follows the Feynman rules as discussed in the previous section, and the numerator of the BW is simply 1 (or i, depending on whose conventions you are following). Thus, the BW form has no particular normalization.
- The BW can be treated as a form factor, modifying the otherwise constant couplings of mesons. If the couplings are fixed at some energy (say, at threshold $s = s_{min}$, or more simply, $s = 0$), the BW can be normalized to be 1 there, and the energy dependence of the BW is used to predict the "running" of the couplings:

$$BW_{FF} = \frac{-m_R^2}{s - m_R^2 - im_R\Gamma_R(s)}. \tag{14}$$

  This was used by Kuhn and Wagner [9] in the context of chiral perturbation theory [10]. Here, the coupling of a gauge boson (*e.g.*, the W from tau decay) to three pseudoscalar particles, in the limit that the momentum of each of the three particles (in the cms frame of all 3) is small compared with Λ_{QCD}, is specified in terms of the pion decay constant (*e.g.*, for 3π, it is V_{ud}/f_π). This is the kinematical regime where the 3-pseudoscalar invariant mass s is only slightly larger than its minimum value (*e.g.*, $s_{min} = (3m_\pi)^2$); far from the peak at $s = m_{a1}^2$. Kuhn and Wagner, and the authors that followed them, assumed that one could extrapolate from s_{min} to larger values by assuming resonance dominance; so they used the BW form given above, normalized so that it equals 1 at $s = 0$. This is the form used in, *e.g.*, TAUOLA [11]. Sometimes you see

$$BW_{FF} = \frac{\Pi(s)}{s - \Pi(s)} = \frac{m_R^2 + im_R\Gamma_R(s)}{s - m_R^2 - im_R\Gamma_R(s)}. \tag{15}$$

Since, as discussed above, the BW is an approximate form for a hadronic resonance, and is really only valid near the peak, this procedure of extrapolating from a small value far out on the tail up to the peak is wholly unjustified. But hey, it's a model; and it works to 10% (in the amplitude) for $\tau \to \rho\nu$ (sheer luck, if you ask me; maybe there's a deep reason, but I have never heard of one).

- In the limit that the resonance is a quasi-stable "fundamental" particle (with fundamental, constant couplings) and can be thought of as being at a fixed mass, it is convenient to normalize the BW to a delta-function:

$$|BW_\delta(s)|^2 = \frac{\sqrt{s}\Gamma(s)/\pi}{(s - m^2(s))^2 + s\Gamma^2(s)} \to \delta(s - m^2) \qquad \text{as} \quad \Gamma(s) \to 0. \tag{16}$$

This arises when one calculates a decay rate

$$\Gamma(s) = \frac{1}{2\sqrt{s}}|M|^2 d\Phi, \tag{17}$$

where the matrix element squared $|M|^2$ contains the propagator BW_{prop} and coupling constants, and the phase space term is $d\Phi$. Let's outline how this happens, for, *e.g.*, $\tau \to K_1\nu$. The phase space factorizes:

$$d\Phi(\tau \to K\pi\pi\nu) = d\Phi(\tau \to K_1\nu) \times d\Phi(K_1 \to K\pi\pi) \times \frac{ds}{2\pi}. \tag{18}$$

This is almost the same as PDG98 eqns 35.10-35.12, except that I put the factor $(2\pi)^4$ into the definition of the phase space (I don't know why they don't). Just as the phase space term factorizes, $|M|^2$ also factorizes:

$$|M(\tau \to K\pi\pi\nu)|^2 = |M(\tau \to K_1\nu)|^2 \times |BW_{prop}(K_1)|^2 \times |M(K_1 \to K\pi\pi)|^2. \tag{19}$$

The process of factorizing $|M|$ into separate *scalar* terms forces us to average over outgoing spin states, so we lose all information on spin correlations in the cascade of decays. But if we do that, we get:

$$\begin{aligned} \Gamma(\tau \to K\pi\pi\nu) &= \frac{1}{2m_\tau}\left[|M(\tau \to K_1\nu)|^2 d\Phi(\tau \to K_1\nu)\right] \\ &\quad \times |BW_{prop}(s)|^2 \left[|M(K_1 \to K\pi\pi)|^2 d\Phi(K_1 \to K\pi\pi)\right]\frac{ds}{2\pi} \\ &= \frac{1}{2m_\tau}\left(|M(\tau \to K_1\nu)|^2 d\Phi(\tau \to K_1\nu)\right)|BW_\delta(s)|^2 \mathcal{B}(K_1 \to K\pi\pi) ds. \end{aligned}$$

We have used the facts that

$$|M(K_1 \to K\pi\pi)|^2 d\Phi(K_1 \to K\pi\pi) = 2\sqrt{s}\Gamma_{K1\to K\pi\pi}(s) = 2\sqrt{s}\Gamma_{tot}(s) \times \mathcal{B}(K_1 \to K\pi\pi), \tag{20}$$

$$|BW_\delta(s)|^2 = |BW_{prop}(s)|^2 \sqrt{s}\Gamma_{tot}(s)/\pi, \tag{21}$$

where $\mathcal{B}$ is the (energy-dependent) branching fraction. (You can take this one step further and define the spectral function $a(s)$). Note that we averaged over all the K_1-spin-dependent correlations, so this form neglects all the physics associated with that.

2.1 Relations between the different forms

Clearly, $BW_{FF} = im_R^2 \times BW_{prop}$ so it is trivial to convert from one to the other. If you have more than one BW (as in K_{1a} and K_{1b}), and have a complex constant fit parameter like β,

$$BW(K_{1a}) + \beta BW(K_{1b}), \tag{22}$$

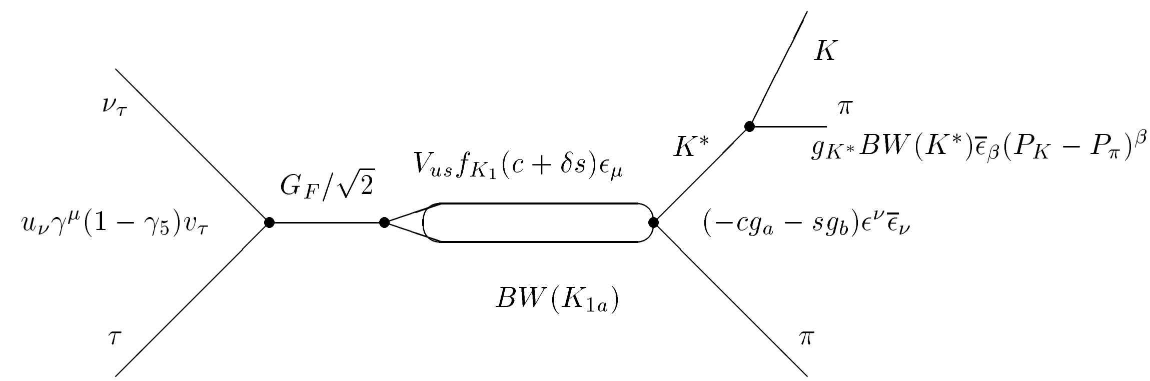
clearly the value of β determined from the fit will be different in for the 2 cases, but trivially related. The delta-normalized BW is a different beast (it's squared). But it allows one to calculate, *e.g.*, the decay rate of the tau to a narrow resonance, in terms of the resonance coupling (the weak decay constant f_{K1}).

For any finite-width resonance, the only reliable way to calculate branching fractions is to numerically integrate the full functional form, setting the fit parameters alternately to 0 or their full value.

3 Application: Decay rate for $\tau \to K\pi\pi\nu$

For easy reference, I include the following relevant section from CBX 98-55:

The Feynman diagram for a typical decay $\tau \to \nu_\tau K_{1a} \to \nu_\tau K^*\pi$, in terms of the couplings defined above, is:



corresponding to the matrix elements:

$$\begin{aligned}
\langle\nu_\tau|J^\mu|\tau\rangle &= (G_F/\sqrt{2})u_\nu\gamma^\mu(1-\gamma_5)v_\tau && (23)\\
\langle K_1|J_\mu|0\rangle &= V_{us}f_{K_1}(c+\delta s)\epsilon_\mu && (24)\\
\langle K^*\pi|H_s|K_1\rangle &= (-cg_a - sg_b)\epsilon^\nu\overline{\epsilon}_\nu BW(K_1) \qquad (\textit{i.e.}, \text{ pure S-wave}) && (25)\\
\langle K\pi|H_s|K^*\rangle &= g_{K^*}\overline{\epsilon}_\beta(P_K - P_\pi)^\beta BW(K^*) && (26)
\end{aligned}$$

Here, ϵ^μ is the polarization vector for the K_1 in some helicity state (we suppress the helicity index, since we will sum over helicities), and $\overline{\epsilon}_\nu$ is the polarization vector for the K^*. Summing over helicities, and requiring the (axial)-vector current to be transverse (*i.e.*, $\epsilon^\mu q_\mu = 0$, that is, $\partial_\mu\epsilon^\mu = 0$, that is, the current is conserved), we have

$$\sum_{helicities}\epsilon^\mu\epsilon^\nu = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) \quad ; \quad \sum_{helicities}\overline{\epsilon}^\nu\overline{\epsilon}^\beta = \left(-g^{\nu\beta} + \frac{P^\nu_{K\pi}P^\beta_{K\pi}}{m^2_{K\pi}}\right). \tag{27}$$

Now, note that the 2nd term in both expressions has the variable q^2 (or $m^2_{K\pi}$) in the denominator, rather than the constant $m^2_{K_1}$ (or $m^2_{K^*}$). Doing the latter would give violations of transversality when $q^2 \neq m^2_{K_1}$, *i.e.*, the vector meson current would not be conserved. In fact, I think that that's the case; requiring transversality is wrong, but it makes the equations a bit simpler, and it is done by many authors (including TAUOLA and Finkmeier & Mirkes). Note also that if the current is transverse, a contribution to the matrix element (Eqn. 26) of the form

$$\langle K\pi|H_s|K^*\rangle = \cdots + g^+_{K^*}\bar{\epsilon}_\beta(P_K + P_\pi)^\beta BW(K^*) \tag{28}$$

can appear if $m_{K\pi} \neq m_{K^*}$. Not only do the above authors neglect such contributions, they even neglect the $P^\nu_{K\pi}P^\beta_{K\pi}/m^2_{K\pi}$ term in the helicity sum, (*i.e.*, they take $\sum \bar{\epsilon}^\nu\bar{\epsilon}^\beta = -g^{\nu\beta}$). This simplifies the equations and is a very good approximation for ρ decay; for K^* decay, this effectively assumes SU(3): $(m^2_K - m^2_\pi)/m_{K^*} \ll 1$).

The Breit Wigners are given by:

$$BW(K_{1a}) = \frac{-1}{q^2 - m^2_{K_1} + i\sqrt{q^2}\Gamma_{K_1}(q^2)} \tag{29}$$

$$BW(K^*) = \frac{-1}{m^2_{K\pi} - m^2_{K^*} + im_{K\pi}\Gamma_{K^*}(m^2_{K\pi})} \tag{30}$$

where the mass dependent widths are discussed below.

Defined in this way, the weak decay constant f_{K_1} has units of mass^2, the strong decay constants g_a and g_b have units of mass, and the strong decay constant g_{K^*} is unitless. Note that I have neglected the appropriate isospin factors; *e.g.*, $g_{K\pi}(K^{*-} \to K^-\pi^0) = \sqrt{1/3}g_{K^*}$ and $g_{K\pi}(K^{*-} \to \bar{K}^0\pi^-) = -\sqrt{2/3}g_{K^*}$. Please put those factors in for your decay mode!

Putting it all together gives:

$$\begin{aligned}\langle K\pi|H_s|K^*\rangle\langle K^*\pi|H_s|K_1\rangle\langle K_1|J^\mu|0\rangle = {} & V_{us}f_{K_1}\cdot(c+\delta s)(-cg_a - sg_b)g_{K^*} \\ & \cdot\left(\frac{\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right)}{q^2 - m^2_{K_1} + i\sqrt{q^2}\Gamma_{K_1}(q^2)}\right) \\ & \cdot\left(\frac{(P_K - P_\pi)_\nu}{m^2_{K\pi} - m^2_{K^*} + im_{K\pi}\Gamma_{K^*}(m^2_{K\pi})}\right)\end{aligned} \tag{31}$$

The total decay rate is obtained in the usual way: $\Gamma = (1/2m_\tau)|M|^2dPS$, where M^μ is the matrix element element given above (averating over tau spins and summing over neutrino spins), and dPS is the integral over 4-body phase space. The phase space integral is quite non-trivial and can only be attempted numerically. However, we can separate out the $\tau \to K_1\nu_\tau$ from the $K_1 \to K\pi\pi$ decays:

$$|M|^2dPS_4(\tau \to K\pi\pi\nu_\tau) = |M_\tau|^2dPS_2(\tau \to K_1\nu_\tau) \times |M_{K_1}|^2dPS_3(K_1 \to K\pi\pi) \times \frac{dq^2}{2\pi} \tag{32}$$

$$M_\tau = L_\mu\langle K_1|J^\mu|0\rangle \tag{33}$$

$$dPS_2(\tau \to K_1\nu_\tau) = \frac{d\Omega_{\nu_\tau}}{32\pi^2}\left(\frac{2p}{m}\right)_\tau = \frac{1}{8\pi}\left(\frac{m^2_\tau - q^2}{m^2_\tau}\right) \tag{34}$$

$$M_{K_1} = \langle K\pi|H_s|K^*\rangle BW_{K^*}\langle K^*\pi|H_s|K_1\rangle \tag{35}$$

$$dPS_3(K_1 \to K\pi\pi) = \frac{d\Omega_\pi}{32\pi^2}\left(\frac{2p}{m}\right)_{K_1}\frac{d\Omega_K}{32\pi^2}\left(\frac{2p}{m}\right)_{K^*}\frac{dm^2_{K\pi}}{2\pi} \tag{36}$$

where all the phase space terms must be evaluated in the rest frame of the decaying particle, and all the interference terms between M_τ and M_{K_1} have been neglected since we will integrate over them.

We define

$$\Gamma_{K\pi\pi} \equiv \frac{1}{2\sqrt{q^2}}|M_{K_1}|^2 dPS_3(K_1 \to K\pi\pi) = \Gamma_{K_{1a}} \times \mathcal{B}(K_{1a} \to K\pi\pi) \qquad (37)$$

and evaluate the matrix elements, using

$$|M_\tau|^2 = L^{\mu\nu} \langle K_1|J_\mu|0\rangle \langle K_1|J^*_\nu|0\rangle \qquad (38)$$

$$L^{\mu\nu} = 2G_F^2 \left(\tau^\mu \nu_\tau^\nu + \nu_\tau^\mu \tau^\nu - g^{\mu\nu}(\tau \cdot \nu_\tau) - i\epsilon^{\mu\nu}_{\alpha\beta}\tau^\alpha \nu_\tau^\beta\right) \qquad (39)$$

$$\langle K_1|J_\mu|0\rangle = V_{us} f_{K_1}(c + \delta s)\epsilon_\mu \qquad (40)$$

$$\langle K_1|J_\mu|0\rangle \langle K_1|J^*_\nu|0\rangle = |V_{us} f_{K_1}(c + \delta s)|^2 \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) \qquad (41)$$

and arrive at the familiar formula:

$$d\Gamma(\tau^- \to K^-_{1a}\nu_\tau) = \frac{G_F^2 V_{us}^2}{16\pi m_\tau^3}(m_\tau^2 - q^2)^2(m_\tau^2 + 2q^2) \cdot \frac{f_{K_1}^2}{q^2}(c + \delta s)^2 \mathcal{W}_{K_{1a}}(q^2)dq^2 \qquad (42)$$

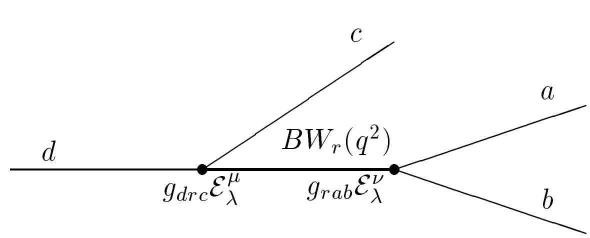
where the normalized Breit Wigner is given by:

$$\mathcal{W}_{K_{1a}}(q^2) = \frac{\sqrt{q^2}\Gamma_{K\pi\pi}(q^2)/\pi}{(m^2_{K_{1a}} - q^2)^2 + (\sqrt{q^2}\Gamma_{K_{1a}}(q^2))^2} \to \delta(m^2_{K_{1a}} - q^2) \qquad \text{as} \quad \Gamma_{K_{1a}} \to 0. \qquad (43)$$

We see the factor $\Gamma_{K\pi\pi}$ in the numerator which comes from the integral over the $K_1 \to K\pi\pi$ decay, and $\Gamma_{K_{1a}}$ in the denominator, which includes *all* decay modes. They are related by the branching fraction $\mathcal{B}(K_{1a} \to K^*\pi)$.

4 Application: Dalitz Plot Matrix Elements

We consider the decay $D \to K\pi\pi$; or, more generally, the 3-body decay of a *spinless* meson $d \to abc$, where a, b, and c are pseudoscalar mesons. We focus on the case where ab resonate to form a resonance r, so we are studying the cascade decay $d \to rc$, $r \to ab$. Here's the Feynman diagram:



4.1 Vector meson intermediate state

Consider first the case where particles a and b resonate as a vector meson r; *e.g.*, $\rho(770)$, $\rho(1450)$, $K^*(892)$, $K^*(1410)$, $K^*(1680)$. The corresponding matrix element is:

$$M = \langle ab|r_\lambda\rangle \, BW_r(q^2) \, \langle cr_\lambda|d\rangle \,, \tag{44}$$

where $q^2 = (P_d - P_c)^2 = (P_a + P_b)^2 = P_{ab}^2 = m_{ab}^2$, λ is the helicity of the vector meson r, and BW_r is the propagator of r. The Feynman rules specify that each vertex at which a vector meson appears must be linear in the polarization vector of the vector meson, $\mathcal{E}_\lambda^\mu$. To get a Lorentz scalar, we must dot $\mathcal{E}_\lambda^\mu$ into some Lorentz vector, and the only other vectors available are the momenta of the pseudoscalar mesons. So the matrix element must take the following forms:

$$\langle cr_\lambda|d\rangle = \left[f_+(q^2)(P_d + P_c)_\mu + f_-(q^2)(P_d - P_c)_\mu\right] g_{drc}\mathcal{E}_\lambda^\mu, \tag{45}$$

$$\langle ab|r_\lambda\rangle = g_{rab}\mathcal{E}_\lambda^{*\nu}(P_a - P_b)_\nu, \tag{46}$$

$$BW_r(q^2) = \frac{-1}{q^2 - m_r^2 - i\sqrt{q^2}\Gamma_r(q^2)}, \tag{47}$$

where f_+ and f_- are strong-interaction form factors which are apriori unknown, g_{drc} and g_{rab} are apriori unknown strong interaction coupling constants which are usually taken to be truly constant (independent of q^2) and which can be obtained from the resonance width, and $\Gamma_r(q^2)$ is the q^2-dependent resonance width.

Also, here we have two such vertices, with an internal vector meson propagator, so we must sum over helicities $\lambda = \pm 1, 0$ before squaring the matrix element:

$$\sum_\lambda \mathcal{E}_\lambda^{*\mu}\mathcal{E}_\lambda^\nu = -g^{\mu\nu} + P_{ab}^\mu P_{ab}^\nu/m^2. \tag{48}$$

Here we come to a subtle but important issue. What do we take for the m^2 in the denominator of the second term: m_{ab}^2 or m_r^2? If the former, this enforces *transversality*: $\mathcal{E}_\lambda^\mu (P_{ab})_\mu = 0$ and $(\sum_\lambda \mathcal{E}_\lambda^{*\mu}\mathcal{E}_\lambda^\nu)(P_{ab})_\mu = 0$ by construction. This enforces a spin-1 current, and is the assumption built in to the Zemach formalism (as we will see below). It corresponds to summing over the helicities $\lambda = \pm 1, 0$, and explicitly neglecting the 4th possible (longitudinal) helicity state. Note that if we wrote $(P_a + P_b)_\nu$ instead of $(P_a - P_b)_\nu$ in the expression for $\langle ab|r_\lambda\rangle$ above, transversality would require the amplitude to be zero. That's why we write $(P_a - P_b)_\nu$ there.

However, if you do this for, *e.g.*, the W boson propagator, the amplitude for $\pi^- \to W^- \to \mu\nu$ would be zero and the pion wouldn't decay. In the Standard Model, we put m_W^2 rather than m_{ab}^2 in the denominator of the second term. Thus, when the W is far off-shell, it has an effective spin-0 component to its current; it can couple to the spin-0 pion. I think most theorists would argue that you should do the same for the vector mesons. In that case, extra terms beyond the Zemach formalism appear; but they go like $(m_a^2 - m_b^2)^2$ and are thus suppressed by isospin for $\rho \to \pi\pi$ or by $SU(3)_f$ for $K^* \to K\pi$.

OK, now we sum over helicities and evaluate the matrix element. The first thing we note is that $(P_d - P_c)^\mu = P_{ab}^\mu$, so that if we require transversality of the vector meson propagator, the term in $f_-(q^2)$ does not constribute.

We still need to know $f_+(q^2)$, but if we approximate $q^2 = m_r^2$ here, f_+ is just a constant factor.

We now have three constant factors: $f_+ g_{drc} g_{rab}$ which can all be absorbed into one unknown constant. More generally, these are really all q^2-dependent. The overall q^2 dependence of all three can be approximated by standard parameterizations, such as the Blatt-Weisskopf form: $g_{drc} \sim (1+(RP)^3)$, where $R =$ a meson radius, $P = \sqrt{q^2}$, and it's to the 3rd power because it's a P-wave decay.

Finally, we must evaluate

$$(P_d + P_c)_\mu (-g^{\mu\nu} + P^\mu_{ab} P^\nu_{ab} / P^2_{ab})(P_a - P_b)_\nu \tag{49}$$

where we are assuming transversality. After a half-page of algebra, we get:

$$(m^2_{bc} - m^2_{ac}) + (m^2_d - m^2_c)(m^2_a - m^2_b)/m^2_{ab}, \tag{50}$$

where we are assuming transversality, so that the denominator of the second term has $P^2_{ab} = m^2_{ab}$. After another page of algebra, we get $-4\vec{p}_a \cdot \vec{p}_c$ (I'm not sure I got all the factors of 2 right), which is the Zemach form. If we *don't* assume transversality, then we must take m^2_r in the denominator of the second term, and the result only reduces to the Zemach form if $m_{ab} = m_r$.

4.2 Mass-dependent width of the intermediate resonance

The q^2-dependent resonance width $\Gamma_{rab}(q^2)$ for the decay $r \to ab$ is given by:

$$\begin{aligned}
\sqrt{q^2}\Gamma_{rab}(q^2) &= \frac{1}{2} |\langle ab | r_\lambda \rangle|^2 dPS_2 &(51)\\
&= \frac{1}{2} g^2_{rab} \left[(P_a - P_b)_\mu (-g^{\mu\nu} + P^\mu_{ab} P^\nu_{ab}/P^2_{ab})(P_a - P_b)_\nu \right] \frac{d\Omega_r}{32\pi^2} \left(\frac{2p}{m}\right)_r &(52)\\
&= \frac{1}{2} g^2_{rab} |\vec{p}_a - \vec{p}_b|^2 \frac{d\Omega_r}{32\pi^2} \left(\frac{2p}{m}\right)_r &(53)
\end{aligned}$$

where dPS_2 is the 2-body phase space for $r \to ab$, Ω_r is the angular phase space for $\hat{p}_a$ in the $r = a$ rest frame, and $(2p/m)_r$ is the usual 2-body phase space term; *e.g.*, for $\rho \to \pi\pi$, $(2p/m)_\rho = \sqrt{1 - 4m^2_\pi / m^2_{\pi\pi}}$. The Zemach result, $|\vec{p}_a - \vec{p}_b|^2$, appears because we took P^2_{ab} in the denominator, *i.e.*, we assumed a transverse vector meson.

For the $\rho \to \pi\pi$ decay, we have a trivial angular integral, and we get:

$$\sqrt{q^2}\Gamma_{\rho\to\pi\pi}(q^2) = \frac{g^2_{\rho\pi\pi} |\vec{p}_\pi|^3}{6\pi\sqrt{q^2}} \tag{54}$$

with $|\vec{p}_\pi| = \sqrt{q^2/4 - m^2_\pi}$. Note that *nowhere* in this expression does the "pole mass" of the ρ appear; it's decay rate does not depend on the pole mass at all. Evaluating this at $m^2_{\pi\pi} = m^2_\rho$ with $\Gamma_{\rho\to\pi\pi}(m^2_\rho) = 150$ MeV gives a numerical value for $g_{\rho\pi\pi}$. There are of course analogous expressions for the K^*.

4.3 Breit Wigners

Remember that a Breit Wigner (BW) lineshape is an approximate "model" for a resonance propagator. It is only "exact", and well-defined, for fundamental particles such as the photon, Z^0, and $W^\pm$. Hadronic resonances, which interact strongly with all other hadrons, are very complex, and the BW approximation for the propagator crops up in a variety of contexts.

If one writes down the Breit Wigner lineshape *precisely* as I have above, then it is easy to add known first radial excitations. For the ρ, we replace BW_ρ with $BW_\rho + \beta BW_{\rho'}$, with $m_{\rho'} = 1.370$ GeV, $\Gamma_{\rho'} = 1.370$ GeV is the "pole mass" width (with $q^2 = m^2_{\pi\pi}$ dependence given exactly as in the case of the ρ), and $\beta = -0.09 \times m^2_{\rho'}/m^2_\rho$. These values are obtained from tau decays; you won't find them in the PDG. The $K^*(1410)$ has not yet been definitively seen in tau decays, although ALEPH claims they see it with the same expression for β as given above.

Note that the *only* place where pole masses appear is in the denominator of the Breit Wigner, Eqn. 47; it is the real part of the resonance self-energy $Re\Pi(q^2)$.

Note that the $\sqrt{q^2}\Gamma_{rab}(q^2)$ which appears in the denominator of the Breit-Wigner is the *total* decay width (imaginary part of the self-energy $Im\Pi(q^2)$), whereas we have been focussing on the partial decay width to final state ab. For the ρ and K^*, the total width is almost completely given by that partial width; for the more massive radial excitations, appropriate branching fractions can be inserted.

4.4 Factorization of the decay rate

If we ignore the fact that the intermediate resonance has several helicity states, the decay rate cleanly factorizes:

$$\begin{aligned} d\Gamma(d \to abc) &= \frac{1}{2m_d}|M(d \to abc)|^2 dPS_3 && (55) \\ &= \frac{1}{2m_d}\left[|M(d \to rc)|^2 dPS_2(d \to rc)\right]\left[|M(r \to ab)|^2 dPS_2(r \to ab)\right] BW_r(q^2)\frac{dq^2}{2\pi} \\ &= d\Gamma_{d\to rc}(q^2)\frac{\sqrt{q^2}\Gamma_{r\to ab}(q^2)/\pi}{q^2 - m_r^2 - i\sqrt{q^2}\Gamma_{r\to ab}(q^2)}dq^2. && (56) \end{aligned}$$

That last term becomes a delta function in the limit $\Gamma_{r\to ab} \to 0$, reducing the expression to an identity for a stable resonance. Of course, the fact that the intermediate resonance *does* does decay, and does have spin, produces the interesting structure in the Dalitz plot, so this factorized form is not terribly useful.

4.5 Scalar meson intermediate state

The scalar resonances are broad and therefore poorly understood. We can have, *e.g.*, $D \to K\sigma$, $\sigma \to \pi\pi$, or $D \to K_0(1430)\pi$, $K_0(1430) \to K\pi$. The σ is so broad, and receives contributions from so many $q\bar{q}$ states, that it is barely a resonance; it's usually modelled as a Breit wignet with a mass of 880 MeV and width of 860 MeV (*cf.*, Tornqvist's Unitarized Quark Model [7]).

If r is a scalar resonance, *e.g.*, $D \to K\sigma$, $\sigma \to \pi\pi$ there are no polarization vectors, so we either get (a) no 4-vectors either, and the matrix element is a constant multiplying the $BW_r(q^2)$; this is S-wave decay; or (b) D-wave decay, with an amplitude given by:

$$\begin{aligned} M &= \langle ab|J^\mu|r\rangle\, BW_r(q^2)\, \langle cr|J_\mu|d\rangle\,, && (57) \\ \langle cr|J_\mu|d\rangle &= \left[f_+(q^2)(P_d + P_c)_\mu + f_-(q^2)(P_d - P_c)_\mu\right] g_{drc}, && (58) \\ \langle ab|J^\mu|r\rangle &= \left[g_+(q^2)(P_a + P_b)^\mu + g_-(q^2)(P_a - P_b)^\mu\right] g_{rab} && (59) \end{aligned}$$

Because the momenta are usually small, the D-wave decay is suppressed relative to the S-wave, and may be neglectable (but this is a quantitative question).

The mass-dependent width of the scalar resonance, assuming S-wave decay only, is simply given by:

$$\sqrt{q^2}\Gamma_{rab}(q^2) = \frac{1}{2}|\langle ab|r\rangle^2 \, dPS_2 \tag{60}$$

$$= \frac{1}{2}g^2_{rab}\frac{d\Omega_r}{32\pi^2}\left(\frac{2p}{m}\right)_r. \tag{61}$$

4.6 Tensor meson intermediate state

We can have, for example, $D \to K_2^*(1430)\pi$, $K_2^*(1430) \to K\pi$.

Here it is clear that we must have D-wave decay for both $d \to rc$ and $r \to ab$, the matrix elements for each decay sequence must be linear in the polarization tensor $\mathcal{E}_\lambda^{\mu\nu}$, and the tensor meson is also assumed to be transverse.

A transverse spin 2 resonance has 5 independent helicity states $\lambda = \pm 2, \pm 1, 0$. Transversality requires that $P_{ab}^\mu \mathcal{E}_{\mu\nu} = P_{ab}^\nu \mathcal{E}_{\mu\nu} = 0$. We need two 4-momenta to dot into the polarization tensor at each vertex. The matrix elements are thus:

$$M = \langle ab|r_\lambda\rangle \, BW_r(q^2) \, \langle cr_\lambda|d\rangle \,, \tag{62}$$

$$\langle cr_\lambda|d\rangle = T_1(q^2)(P_d + P_c)_\mu (P_d + P_c)_\nu \mathcal{E}_\lambda^{\mu\nu}, \tag{63}$$

$$\langle ab|r_\lambda\rangle = T_2(q^2)(P_a - P_b)_\mu (P_a - P_b)_\nu \mathcal{E}_\lambda^{*\mu\nu}, \tag{64}$$

Here's where it gets really messy. The expression for $\sum_\lambda \mathcal{E}_\lambda^{*\mu\nu}\mathcal{E}_\lambda^{\alpha\beta}$ is very nasty, and I'm unwilling to TeX it up (but see Gasiorowicz [3] or Pilkuhn [12]).

In the Zemach formalism, we get:

$$M = T_1(q^2)BW_r(q^2)T_2(q^2)|\vec{p}|^2 d_{00}^{(2)}(\cos\theta), \tag{65}$$

$$\sqrt{q^2}\Gamma_{r\to ab}(q^2) = \frac{1}{2}\left|T_1(q^2)BW_r(q^2)T_2(q^2)|\vec{p}|^2\right|^2 |d_{00}^{(2)}(\cos\theta)|^2 \frac{d\Omega_r}{32\pi^2}\left(\frac{2p}{m}\right)_r \tag{66}$$

$$= \left|T_1(q^2)BW_r(q^2)T_2(q^2)\right|^2 \frac{4|\vec{p}|^5}{15\pi\sqrt{q^2}}. \tag{67}$$

Here again, the decay rate does not depend on the pole mass at all.

5 Form Factors

Simple field theory assumes that all particles are point-like. In nuclear physics, the effect of the finite extent of nucleii is a much-studied subject. Mesons, like nucleii, have finite extent, and the net effect on decay rates and scattering cross-sections can be parameterized by form factors. These can be thought of as modifications to the coupling constants which are otherwise assumed to be constant; finite size effects give a mass dependence to these couplings (*e.g.*, to the strong coupling g_{rab} in Eqn. 47 above). These finite size effects are *over and above* the resonant effects discussed above that give rise to Breit Wigners.

The finite size of a parent particle can be described by a "strong-charge distribution", which we can crudely model as a 3D Gaussian with parameter R:

$$\psi(\vec{r}) = \frac{1}{(\pi R^2)^{3/4}} e^{-r^2/2R^2}, \tag{68}$$

so that the "radius" can be interpreted as $\sqrt{< r^2 > / < r^0 >} = \sqrt{6}/2R$. Then the form-factor $F(p)$ is its Fourier transform with respect to the (magnitude of the) probe momentum transfer p:

$$F(p) \propto e^{-(Rp)^2/2} \approx 1 - \frac{(Rp)^2}{2} + \mathcal{O}(Rp)^4. \tag{69}$$

If the decay of such a particle proceeds with non-zero angular momentum, the Fourier transform is a bit more complicated. A relativistic treatment is more complicated still.

A model of finite size effects that is often used in non-relativistic nuclear physics is to think of the daughter particles as rattling around in a common potential, which you can model as a spherical well with a barrier (due, eg, to the coulomb potential between the daughters when they are outside the well; but it does not need to have anything to do with the coulomb force, it's just something to hold the parent meson together. In our case, it's clearly a strong-force effect). The daughters have to tunnel through the barrier in order for the parent particle to decay.

This model is treated in Blatt & Weisskopf's classic text on nuclear physics [12]. You may recall from your advanced QM classes (*cf.*, Gasiorowicz, or Schiff) that the asymptotic wave functions in a spherical well potential are spherical Hankel functions. In good old non-relativistic phase-shift scattering theory, the daughter mesons have phase-shifted Hankel wave functions in the asymptotic limit $r < R$, where R is the minimum radius where the potential is negligible.

B&W use this picture to calculate the barrier penetration factor (which, in our context, is just the amplitude form factor for the decay). The Blatt-Weisskopf barrier penetration factor [12] is given by the logarithmic derivative of the Hankel wave functions evaluated at $r = R$ (standard procedure in evaluating scattering phase shifts off of spherical potentials). It amounts to a measure of the suppression of the process at non-zero angular momentum due to the centifugal barrier holding the daughter particles together. For $L = 1$ (as in $V \to PP$), it amounts to replacing, *e.g.*, $g^2|\Delta p|^2$ in the expression for M in Eqn. 46 with $g'^2(\Delta p)^2R^2/(1 + (\Delta p)^2R^2)$. Thus, there is an extra momentum dependence due to the centifugal barrier. In terms of the decay width of a vector resonance decaying to two pseudoscalars ($r \to ab$), its energy dependence is given by:

$$m_{ab}\Gamma(p_a) = m_r\Gamma_r \cdot \frac{m_r}{m_{ab}} \left(\frac{p_a}{p_r}\right)^3 \cdot \frac{1 + (Rp_r)^2}{1 + (Rp_a)^2}, \tag{70}$$

where Γ_R is the width at peak ($m_{ab} = m_r$), p_a is the momentum of particle a or b in the ab center-of-mass, p_r is p_a when $m_{ab} = m_r$, and R is the effective radius of resonance r. The last term in Eqn. 70 is the Blatt Weisskopf barrier penetration factor.

The barrier penetration factor is of course approximate; we don't know the true nature of the meson potential, and in any case it isn't relativistic.

I believe that these effects are real, and that these additional form factors, despite being crude non-relativistic approximations, should rightly be included in the description of any high-precision measurement involving mesonic dynamics. Unfortunately, since it is approximate, it introduces a model dependence; but that should be expected.

The effect is not negligible, even for the K^*, being a 10% effect on the width at $(m_{K^*} - \Gamma_{K^*})$. I would expect that we would be sensitive it it in, *e.g.*, the $D \to K\pi\pi$ Dalitz plot, and further, that a plot of fit likelihood vs R (perhaps from a succession of fits, stepping through fixed R) should reveal a minimum somewhere sensible.

References

[1] See the analysis of $\tau \to (K\pi\pi)\nu$ by Ilya Kravchenko.

[2] See the analysis of the $D^0 \to K^-\pi^+\pi^0$ Dalitz plot by Tim Bergfeld.

[3] See, for example, section 7.3 of "Introduction to High Energy Physics", D. Perkins Addison-Wesley (1972); section 7.8 of "Modern Quantum Mechanics", J. Sakurai, Addison-Wesley (1985); section 19 of "Quantum Mechanics", L. Schiff, McGraw-Hill (1968); chapter 23 of "Quantum Physics", S. Gasiorowicz, John Wiley (1996); or chapter X of "Quantum Mechanics", A. Messiah, John Wiley (1976).

[4] See, for example, section 7.3 of Halzen & Martin eqn 6.87). See, for example, section 7.3 of "An Introduction to Quantum Field Theory", M. Peskin and D. Schroeder, Addison-Wesley (1995); eqn 6.87 of "Quarks and Leptons", F. Halzen and A. Martin, Wiley (1984); eqn. 10.113 of "Introduction to Elementary Particles", D. Griffiths, Wiley (1987).

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[12] See section VIII.5 of "Theoretical Nuclear Physics", J. Blatt and V. Weisskopf, Dover, 1979. Also see eqn. 4-9.16 of "Relativistic Particle Physics", H. Pilkuhn, Springer-Verlag (1979).